Dirac Delta Function and Unit Impulse Function

[11] 7.6.7 Before we begin investigating this unique function, we must acknowledge that it is an operator and all operators have instructions; without the instructions we cannot perform said operations.

Unit Impulse Function

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \dfrac{1}{2a}, & t_0 - a \le t < t_0 + a\,, \\ 0, & t \ge t_0 + a \end{cases} \qquad a > 0, t_0 > 0.$$

$$\int_0^{\infty} \delta_a(t-t_0) = 1.$$

The Dirac Delta Function: Defined by taking the limit of the **Unit Impulse** function. That is,

$$\lim_{a\to 0} \delta_a(t-t_0) = \delta_0(t-t_0) \equiv \delta(t-t_0).$$

$$\text{[i]}\ \ \delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \ne t_0 \end{cases}, \qquad \text{[ii]}\ \int_0^{\infty} \delta(t-t_0)\,dt = 1.$$

FOCUS DEFINITION Dirac Delta Integration

$$\int_0^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0)$$

Sifting Property: When an object in the integrand is simply evaluated at the constant within the integrand leading to no formal arithmetic required.

$$\int_0^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0)$$

[FE] This formula will be introduced in advance physics related (ENGR) courses junior year.

NOTE All of this is just rules and instructions based on definitions. Don't try to understand anything as it is about as productive as understanding why the letter A has the shape and sound it does—that is, someone defined it that way.

[TM] This is a function that applies in special cases and there is no need to contemplate the meaning behind it as it is simply a definition.

Laplace Transform: Dirac Delta Function

Solve the second order ODE using method of Laplace Transform,

$$y'' + 2y' = \delta(t-1), \qquad y(0) = 0, \qquad y'(0) = 1.$$

We begin by apply the Laplace transform

$$y'' + 2y' = \delta(t-1) \;\Rightarrow\; \mathcal{L}\{y'' + 2y' = \delta(t-1)\}$$

NOTE The Laplace Operator is a linear transformation.

DEFINITION **Linear Transformation – Laplace Operator is a Linear Transformation**

$$\mathcal{L}\{f(\alpha t) + g(\beta t)\} = \mathcal{L}\{f(\alpha t)\} + \mathcal{L}\{g(\beta t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$$

$$\Rightarrow\; \mathcal{L}\{y''\} + \mathcal{L}\{2y'\} = \mathcal{L}\{\delta(t-1)\} \;\Rightarrow\; \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} = \mathcal{L}\{\delta(t-1)\}$$

Now that we are at the linear transformation, we need the Laplace Transformation information.

THEREOM Laplace Transform

$$\mathcal{L}\{y\} = Y(s), \qquad \mathcal{L}\{y'\} = sY(s) - y(0), \qquad \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0), \ldots$$

or

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

$$f, f', \ldots, f^{(n-1)} \text{ are continuous on } \mathbb{R}^+ \cup \{0\} \equiv [0,\infty).$$

$$\Rightarrow\; [s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] = \mathcal{L}\{\delta(t-1)\}$$

Dirac Delta Laplace Transform

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

$$\Rightarrow\; [s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] = e^{-st_0}$$

$$\Rightarrow\; s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) = e^{-st_0} \text{ [solve for } Y(s)]$$

$$\Rightarrow\; s^2Y(s) + 2sY(s) - sy(0) - y'(0) - 2y(0) = e^{-st_0}$$

$$\Rightarrow\ \ s^2Y(s)+2sY(s)=e^{-st_0}+\big(sy(0)+y'(0)+2y(0)\big)$$

$$\Rightarrow\ \ s^2Y(s)+2sY(s)=e^{-st_0}+\big((s)(0)+(1)+2(0)\big)$$

$$\Rightarrow\ \ Y(s)(s^2+2s)=e^{-st_0}+1$$

$$\Rightarrow\ \ Y(s)=\frac{e^{-s(1)}+1}{s^2+2s}=\frac{e^{-s}+1}{s^2+2s}.$$

Now, the tricky part: We have to arrange this in a form that fits a pattern that matches a preexisting Laplace Inverse. We go to the table of Laplace Inverses (usually in the back of the textbook).

Laplace Inverse

$$\mathcal{L}^{-1}\{F(s)\}=f(t),\qquad \mathcal{L}^{-1}\{Y(s)\}=y(t).$$

Examine the possible alterations to the existing function before making any final decisions.

$$Y(s)=\frac{e^{-s}+1}{s^2+2s}=\frac{e^{-s}}{s^2+2s}+\frac{1}{s^2+2s}=e^{-s}\frac{1}{s^2+2s}+\frac{1}{s^2+2s}$$

→TYPO TO BE CORRECTED STARTING HERE← [everything above is correct]

$$=e^{-s}\frac{1}{(s+1)^2-2}+\frac{1}{(s+1)^2-2}=e^{-s}\frac{1}{(s+1)^2-\sqrt{2}^2}+\frac{1}{(s+1)^2-\sqrt{2}^2}$$

From the Laplace Inverse Table

$$\mathcal{L}^{-1}\left\{\frac{k}{(s-a)^2-k^2}\right\}=e^{at}\sinh kt$$

*BIG TYPO ↘

$$\mathcal{L}^{-1}\{e^{-as}F(s)\}=f(t-a)\mathcal{U}(t-a)$$

$$\Rightarrow\ \ \mathcal{L}^{-1}\left\{Y(s)=\frac{e^{-s}+1}{s^2+2s}\right\}\ \ \Rightarrow\ \ y(t)=\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{(s+1)^2-\sqrt{2}^2}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2-\sqrt{2}^2}\right\}$$

$$\Rightarrow\ \ y(t)=\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{(s+1)^2-\sqrt{2}^2}\right\}+\frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s-(-1))^2-\sqrt{2}^2}\right\}$$

$$\Rightarrow \; y(t) = f(t-1)\mathcal{U}(t-1) + \frac{1}{\sqrt{2}}\, e^{-t}\sinh\left(\sqrt{2}t\right)$$

$$= \frac{1}{\sqrt{2}}\, e^{-(t-1)}\sinh\left(\sqrt{2}(t-1)\right)\mathcal{U}(t-1) + \frac{1}{\sqrt{2}}\, e^{-t}\sinh\left(\sqrt{2}t\right).$$

[FC] To prove that the answer is in fact the correct answer, take it and then compute the derivatives and plug them into $y'' + 2y' = \delta(t-1)$ (the original equation); if the equation balances—that is, $0 = 0$, then you know you did it correctly. Hence, you do not need ChatGPT or a teacher's solution manual and you have effectively done mathematics—not puzzle solving. The time to do this, is too lengthy at this moment so we will check the answer in the back of the book but before checking the answer, make sure you take the time to acknowledge the steps of reverse engineering and proving the solution you provided that outputted the answer, is in fact the correct answer.

NOTE This was an hour and half of work and it may not be right. We check the answer in the back of the book now.

The books answer is

$$y = \frac{1}{2} - \frac{1}{2}e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1).$$

This does not mean our answer is incorrect, it may be in a different form is all. Our answer is similar but not identical. Recall the hyperbolic trigonometric function

$$\sinh(t) = \frac{e^t - e^{-t}}{2}.$$

For this question, I will leave it to you to verify whether I made a typo or if

$$\frac{1}{2} - \frac{1}{2}e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1) = \frac{1}{\sqrt{2}}\, e^{-(t-1)}\sinh\left(\sqrt{2}(t-1)\right)\mathcal{U}(t-1) + \frac{1}{\sqrt{2}}\, e^{-t}\sinh\left(\sqrt{2}t\right).$$

Check

$$\mathcal{L}^{-1}\left\{\frac{k}{(s-a)^2 - k^2}\right\} = e^{at}\sinh kt = e^{at}\left[\frac{e^{kt} - e^{-kt}}{2}\right] \;\Rightarrow\; \sinh\left(\sqrt{2}t\right) = \frac{e^{\sqrt{2}t} - e^{-\sqrt{2}t}}{2}$$

There answer might be wrong or mine may be wrong or they are both the same. You check on your own without ChatGPT or solution manuals. Do not give up! We will come back to this question tomorrow and go back through it to be sure and finalize the answer and verify if it is correct after you all review it as part of the researchers.

→ ALL ABOVE THIS LINE IS INCORRECT → Corrected version below (reference both)

Starting with

$$Y(s) = \frac{e^{-s}+1}{s^2+2s} = \frac{e^{-s}+1}{s^2+2s+1-1} = \frac{e^{-s}+1}{(s+1)^2-1} = \frac{e^{-s}+1}{\left(s-(-1)\right)^2-1}$$

From the Laplace Inverse Table

$$\mathcal{L}^{-1}\{Y\} = y, \qquad \mathcal{L}^{-1}\left\{\frac{k}{(s-a)^2-k^2}\right\} = e^{at}\sinh kt$$

$$\mathcal{L}^{-1}\{e^{-bs}F(s)\} = f(t-b)\mathcal{U}(t-b)$$

Note that the "a" and "b" are both a in the formula book but they are different a's here.

$$Y(s) = \frac{e^{-s}+1}{s^2+2s} = \frac{e^{-s}+1}{s^2+2s+1-1} = \frac{e^{-s}+1}{(s+1)^2-1} = \frac{e^{-s}+1}{\left(s-(-1)\right)^2-1}$$

$$= e^{-s}\frac{1}{\left(s-(-1)\right)^2-1} + \frac{1}{\left(s-(-1)\right)^2-1} = e^{-s}Y(s) + Y(s)$$

$$\Rightarrow \ \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{e^{-s}Y(s)\} + \mathcal{L}^{-1}\{Y(s)\}$$

$$\Rightarrow \ \mathcal{L}^{-1}\{e^{-s}Y(s)\} + \mathcal{L}^{-1}\{Y(s)\} = y(t-1)\mathcal{U}(t-1) + e^{-t}\sinh t$$

$$= e^{-(t-1)}\sinh(t-1)\,\mathcal{U}(t-1) + e^{-t}\sinh t$$

Hyperbolic Sine Function Operation

$$\sinh(t) = \frac{e^t - e^{-t}}{2}, \qquad \exp\{ax\} = e^{ax} = \sum_{k=0}^{\infty}\frac{a^k x^k}{k!} = 1 + ax + \frac{a^2x^2}{2} + \cdots$$

$$= e^{-(t-1)}\left[\frac{e^{t-1} - e^{-(t-1)}}{2}\right]\mathcal{U}(t-1) + e^{-t}\left[\frac{e^t - e^{-t}}{2}\right]$$

$$= \left[\frac{e^{(t-1)-(t-1)} - e^{-(t-1)-(t-1)}}{2}\right]\mathcal{U}(t-1) + \frac{e^{t-t} - e^{-t-t}}{2}$$

$$= \left[\frac{e^0 - e^{-2(t-1)}}{2}\right]\mathcal{U}(t-1) + \frac{e^0 - e^{-2t}}{2}$$

$$= \left[\frac{1 - e^{-2(t-1)}}{2}\right]\mathcal{U}(t-1) + \frac{1 - e^{-2t}}{2}.$$

The books answer is

$$y = \frac{1}{2} - \frac{1}{2}e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t-1).$$

