Is the polynomial **vector** $f(t) = t^2 - t$ **in the span** of $\mathcal{P} = \{t, t^2, t + t^2\}$?

When you are solving mathematically based problems, you are given the instructions within the question. You simply follow the instructions. In this case, the instructions come from relating the English to the math. "As one sees the forest through the trees, one sees the math through the physics."

Coming from the **Definition of Span and Solution Sets**: In this case, "is $f(x)$ in the span" means to set the span equal to $f(x)$ and see if there are unique, infinite, or no solutions.

From the question, the solution lies within the keyword's "**vector**" and "**in the span**." In your textbook, these words would point to a definition that would point to the following solution.

Thus, (according to the span theorem/definition)

$$\text{span}\,\mathcal{P} = f(t) \;\Rightarrow\; \text{span}\{t, t^2, t + t^2\} = t^2 - t$$

$$\Rightarrow\; tx_1 + t^2x_2 + (t + t^2)x_3 = t^2 - t = (1)t^2 + (-1)t + (0)(1)$$

$$\Rightarrow\; tx_1 + t^2x_2 + (t + t^2)x_3 = \begin{bmatrix} t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow\; \begin{bmatrix} t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}x_1 + \begin{bmatrix} t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}x_2 + \begin{bmatrix} t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}x_3 = \begin{bmatrix} t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow\; \begin{bmatrix} t^2 & t & 1 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}x_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}x_2 + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}x_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) \text{ [bubble matrix]}$$

Note: You cannot move this to the right because the dimensions would conflict. I.e., you have a $(1 \times 3)(3 \times 1)(1) = 1 \times 1$. Matrix distribution rules that must be followed. The function, $f(x)$ is a 1×1 scalar function. We just undid the product.

$$\Rightarrow\; \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}x_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}x_2 + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}x_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)_{\mathcal{P}_3}$$

$$\Rightarrow\; \left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)_{\mathcal{P}_3} \;\Rightarrow\; \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]_{\mathcal{P}_3}$$

In Wolfram Alpha, RREF {{0,1,1,1},{1,0,1,-1},{0,0,0,0}}

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]_{\mathcal{P}_3} \Rightarrow \begin{array}{c} x_1 + x_3 = -1 \\ x_2 + x_3 = 1 \\ x_3 = x_3 \text{ free} \end{array} \Rightarrow [t^2 \quad t \quad 1]\left[\mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} x_3\right]$$

From the matrix, we see that there are an infinite number of solutions. Thus, $\mathbf{f} \in \text{span}\,\mathcal{P}$. And the following is the solution.

[RE] If the solution is correct, then the solution found should be able to balance the initial equation.

$$\therefore \mathbf{x} = [t^2 \quad t \quad 1][-1 - x_3, 1 - x_3, x_3] = t^2(-1 - x_3) + t(1 - x_3) + x_3$$

$$\Rightarrow \ \mathbf{f} = (-1 - x_3, 1 - x_3, x_3)_{\mathcal{P}_3}.$$

Recall

$$\mathbf{f} \equiv f(x) \quad \Leftrightarrow \quad \begin{bmatrix} -1 - x_3 \\ 1 - x_3 \\ x_3 \end{bmatrix}_{\mathcal{P}_3} \equiv t^2(-1 - x_3) + t(1 - x_3) + x_3$$

We want to verify our work is correct. If it is correct, then the above answer should mean that the span equals the function. I.e.,

$$\text{span}\{t, t^2, t + t^2\} = t^2 - t \ \Leftrightarrow \ tx_1 + t^2x_2 + (t + t^2)x_3 = t^2 - t$$

Plug $\begin{bmatrix} -1 - x_3 \\ 1 - x_3 \\ x_3 \end{bmatrix}_{\mathcal{P}_3}$ this in for x_1, x_2, x_3. If it helps to change $x_3 = s$ for a free variable, we can do that. Let $\begin{bmatrix} -1 - x_3 \\ 1 - x_3 \\ x_3 \end{bmatrix}_{\mathcal{P}_3} = \begin{bmatrix} -1 - s \\ 1 - s \\ s \end{bmatrix}_{\mathcal{P}_3}$ for standard parametric solution.

Now, to check our work

$$\text{span}\{t, t^2, t + t^2\} = t^2 - t$$

$$\Rightarrow \ tx_1 + t^2x_2 + (t + t^2)x_3 = t(-1 - s) + t^2(1 - s) + (t + t^2)(s)$$

$$= -t - ts + t^2 - t^2s + ts + t^2s = t^2 - t.$$

Therefore, $f(x)$ is in the span of the provided set, and the solution to the system of equations is

$$f(x) = t^2(-1 - s) + t(1 - s) + s.$$

How to structure this on an exam for max credits and time reduction

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Is the polynomial **vector** $f(x) = t^2 - t$ **in the span** of $\mathcal{P} = \{t, t^2, t + t^2\}$?

By definition,

$$\operatorname{span}\mathcal{P} = f(x) \;\; \Rightarrow \;\; \operatorname{span}\{t, t^2, t + t^2\} = t^2 - t.$$

Thus,

$$\begin{bmatrix} t^2 & t & 1 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$$

$$\Rightarrow \left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)_{\mathcal{P}_3} \;\; \Rightarrow \;\; \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]_{\mathcal{P}_3}.$$

Because there are infinite solutions, we see that f is in the span.

The above is all you really need to write for the exam, but you may have more arithmetic based on your skill set at the time of the exam.