The Remaining Case of the Perfect Cuboid

Arithmetic exclusions and a geometric formulation

Author Jonathan David
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Abstract. We derive parity and divisibility restrictions, exclude infinite families of odd edges, and give an exact divisor criterion for a perfect cuboid. We report an exhaustive bounded search and an equivalent geometric formulation, then state the unresolved condition for future work.

1. The conjecture and squared-length identities

Let a,b,c be positive integer edges, d_{ab},d_{ac},d_{bc} the face diagonals, and D the space diagonal. An Euler brick has integer face diagonals. It is a perfect cuboid if D is also integral [1].

Conjecture. Every Euler brick has an irrational space diagonal.

For a positive integer N, if $\sqrt{(N)} = u/v$ in lowest terms, then $v^2 \mid u^2$, forcing $v = 1$. Thus the conjecture is equivalent to nonexistence of a perfect cuboid. The Pythagorean theorem gives

$$d_{ab}^2 + d_{ac}^2 + d_{bc}^2 = 2D^2,$$
$$a^2 + b^2 + c^2 + d_{ab}^2 + d_{ac}^2 + d_{bc}^2 = 3D^2.$$

Each squared edge occurs twice in the first sum. Adding $a^2 + b^2 + c^2 = D^2$ gives the second identity.

2. Parity and divisibility

Write $g = gcd(a,b,c)$. Each integer diagonal is divisible by g because its square is divisible by g^2. Dividing all lengths by g therefore preserves the integer diagonal conditions and the rationality of D. We may restrict attention to primitive cuboids, for which $g = 1$.

• With two odd edges, their common face has squared diagonal $2 \bmod 4$, so it cannot have an integer diagonal. The space diagonal squared is also $2 \bmod 4$.

• With three odd edges, each squared face diagonal is $2 \bmod 4$, and $D^2 \equiv 3 \bmod 4$.

• With all even edges, primitive reduction applies.

• With one odd edge, an Euler brick has two odd face diagonals and one even face diagonal. Its space diagonal must be odd if integral.

Equation the face-diagonal identity independently excludes one or three odd face diagonals when D is integral. All even face diagonals require all even edges: even face diagonals force matching edge parities, and two odd edges are impossible in an Euler brick.

Let a be odd. An even edge congruent to $2 \bmod 4$ would give a squared face diagonal congruent to $5 \bmod 8$ with a, so $4 \mid b,c$. Write $b = 2^r u$, $c = 2^s v$, with u,v odd and $2 \le r \le s$.

The integer $u^2 + 2^{2(s-r)}v^2$ must be a square. For $s = r$ it is $2 \bmod 4$, and for $s = r + 1$ it is $5 \bmod 8$. Thus $s \geq r + 2$.

Exactly two edges in a primitive Euler brick are divisible by 3. Two edges indivisible by 3 would give a squared face diagonal congruent to $2 \bmod 3$, and primitivity excludes three edges divisible by 3. At least one edge is divisible by 5. Otherwise two squared edges would have the same nonzero residue modulo 5, and their sum would be 2 or $3 \bmod 5$, neither a square.

Proposition 1. *The four diagonal square congruences admit simultaneous solutions modulo every power of two, within the surviving edge parity pattern.*

Proof. The edges $(1,12,16)$ give squared face diagonals $145,257,400$ and squared space diagonal 401. The even edges have valuations $2,4$. The number 400 is a square and the other three numbers are $1 \bmod 8$. For $N \equiv 1 \bmod 8$, a root exists modulo 8. If an odd x is a root modulo 2^k, $k \geq 3$, the squares of x and $x + 2^{k-1}$ differ by $2^k \bmod 2^{k+1}$. Exactly one is a root modulo 2^{k+1}. Induction proves the claim. □

The example satisfies the congruences and fails the exact square conditions for $145,257,401$. Solvability of the system modulo powers of two therefore leaves the exact equations unresolved.

3. Divisor formulation

For odd a, define

$$S_a = \{(a^2/d - d)/(2) : d \mid a^2, 0 < d < a\}.$$

A positive integer x belongs to S_a exactly when $a^2 + x^2 = h^2$ for an integer h. To see this, factor $a^2 = (h - x)(h + x)$. Conversely, the two positive factors $d, a^2/d$ are odd, so their half-sum and half-difference give h,x.

A primitive perfect cuboid is equivalent to a solution of

$$b,c,e \in S_a,\ b^2 + c^2 = e^2,\ \gcd(a,b,c) = 1.$$

Here $e = d_{bc}$. The memberships give integer diagonals for the faces joining a to b,c, and for the right triangle with legs a,e, whose hypotenuse is D.

Proposition 2. *The odd edge of a primitive perfect cuboid is neither 1 nor a prime power.*

Proof. The set S_1 is empty. Let $a = p^k$ for an odd prime p. If $x \in S_a$ is coprime to p, the factors $h - x, h + x$ are powers of p with difference $2x$ coprime to p. The smaller is 1, so $x = (a^2 - 1)/2$ is the unique such candidate.

If $p \nmid e$, at least one of b,c is also coprime to p by primitivity. Uniqueness forces that edge to equal e, although $e > b,c$. If $p \mid e$, then $p \mid b$ would imply $p \mid c$, contradicting primitivity. The same reasoning with the edges exchanged shows both are coprime to p. Hence $b = c$, making $e = b\sqrt{2}$ irrational. □

Proposition 3. *The odd edge cannot have exactly two distinct prime factors both congruent to 3 mod 4.*

Proof. Write $a = PQ$, where $P = p^r < Q = q^s$ and p,q are distinct primes congruent to $3 \bmod 4$. The largest element of S_a is $M = (a^2 - 1)/2$, from $d = 1$. Each other element has $d \geq 3$ and is less than $a^2/6$. If $e = M$, then $b,c < e$ give $e = \sqrt{(b^2 + c^2)} < b + c < a^2/3 < M$. Thus $e \neq M$.

Neither prime divides e. Otherwise nonresiduosity of -1 modulo that prime would force it to divide both even edges, violating primitivity. Therefore $D - e, D + e$ are coprime factors of P^2Q^2. The only coprime pairs are $(1, P^2Q^2)$ and (P^2, Q^2), giving M and $e = (Q^2 - P^2)/2$, respectively. These are the only two members of S_a coprime to a. Since $b, c < e$, each shares a prime factor with a, and by primitivity they share distinct primes.

Relabel so $p \mid b$ and $q \nmid b$, and write $f = \sqrt{(a^2 + b^2)}$. The factor Q^2 lies entirely in one of $f - b, f + b$ since $q \nmid 2b$. It lies in the larger factor since $Q^2 > P^2$. Also $p \mid f, b$, so p divides both factors. Hence $f + b \geq pQ^2$ and $f - b \leq P^2/p$. Subtracting gives $2b \geq pQ^2 - P^2/p > Q^2 - P^2 = 2e$, contrary to $b < e$. □

4. Computational check

For each positive odd a through $30{,}000{,}000$, we enumerated S_a and tested each unordered pair b, c surviving the proved restriction on powers of two. We tested $b^2 + c^2$ for an integer square, imposed primitivity, and tested $a^2 + b^2 + c^2$ for an integer square. The second interval also used a square-residue filter modulo 63. Square-root estimates were corrected and checked by integer multiplication. An independent enumeration agreed through $a = 5{,}000$.

Odd-edge interval	Primitive Euler bricks	Perfect cuboids
Through $10{,}000{,}000$	1,167	0
Above $10{,}000{,}000$ through $30{,}000{,}000$	730	0
Total	1,897	0

The cutoff applies only to the odd edge. The divisor enumeration covers all corresponding even edges. Robert D. Matson reported an exhaustive exclusion below an odd edge of 2.5×10^{13} [2]. The computation here independently checks a smaller range.

5. Circumscribed sphere and rational coordinates

The center of an inscribed rectangular cuboid is the center of the sphere. Indeed, if the cuboid center is the origin, equating distances from the sphere center to opposite vertices makes the center orthogonal to three independent vertex directions, forcing it to be the origin. Opposite vertices are antipodal, so $D = 2R$ with R the sphere radius.

Set $x = a/D$, $y = b/D$, $z = c/D$. Then

$$x^2 + y^2 + z^2 = 1, \ (d_{bc}, d_{ac}, d_{ab})/D = (\sqrt{(1 - x^2)}, \sqrt{(1 - y^2)}, \sqrt{(1 - z^2)}).$$

The conjecture is equivalent to every positive rational point on this sphere having at least one irrational coordinate-plane projection length. Clearing denominators proves the converse. The rational-point viewpoint is established in the literature [3]. Scaling R leaves the normalized point unchanged. A single finite perfect cuboid would yield arbitrarily large examples by integer scaling.

One may parametrize the three circle conditions by writing $x = 2u/(1 + u^2)$, $\sqrt{(1 - x^2)} = (1 - u^2)/(1 + u^2)$, with rational $0 < u < 1$, and similarly for y, z. Substitution into the sphere equation gives one rational equation in three parameters. An infinite descent would require a map that preserves positivity and all rationality conditions while strictly decreasing a

positive integer measure, such as the primitive space diagonal. Such a map remains to be found.

A related elliptic-curve formulation fixes $t = b/a > 0$, for which $1 + t^2$ is a rational square, and sets $X = (c/a)^2$. The hypothetical cuboid gives a point on

$$Y^2 = (X + 1)(X + t^2)(X + 1 + t^2).$$

The converse requires X to be a positive rational square and each of the three factors individually to be a rational square. The product condition alone is weaker. The evident point at $X = 0$ corresponds to a zero edge.

6. For a future mathematician

The remaining task is to exclude or exhibit the divisor condition above. Consider every odd a with at least two distinct prime factors, excluding those with exactly two distinct prime factors both congruent to $3 \bmod 4$. For each such a, ask whether S_a contains b,c,e with $b^2 + c^2 = e^2$ and $\gcd(a,b,c) = 1$. The proved parity and divisibility conditions can be imposed throughout.

A proof that every remaining S_a lacks such a triple would establish that no perfect cuboid exists. One explicit triple would settle the problem in the other direction. To verify a proposed example, give the three edges (a,b,c), the three integer face diagonals, and the integer D satisfying $D^2 = a^2 + e^2$. The argument in this paper leaves this exact existence question open.

Educational note on AI

ChatGPT assisted with drafting arguments and running a bounded computation. It did not resolve the conjecture. Fluent output can hide a missing implication, and an exhaustive finite search stops at its cutoff. Students should build the ability to work through arguments independently, explain each inference in their own words, and disclose AI assistance where required. Passing off generated solutions as one’s own defeats that purpose. A trained mathematician who can verify a proof independently of AI should scrutinize any claimed resolution before it is presented as established.

References

[1] E. W. Weisstein, *Perfect Cuboid*, Wolfram MathWorld. https://mathworld.wolfram.com/PerfectCuboid.html.

[2] R. D. Matson, *Results of computer search for a perfect cuboid*, 2015. https://unsolvedproblems.org/S58.pdf.

[3] *Notices of the American Mathematical Society*, section 4.2, Perfect cuboids, 2021. https://doi.org/10.1090/noti2335.